Spin echo without an external permanent magnetic field

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The spin echo techniques aim at the elimination of the effect of a random magnetic field on the spin evolution. These techniques conventionally utlize the application of a permanent field which is much stronger than the random one. The strong field, however, may also modify the magnetic response of the medium containing the spins, thus altering their "natural" dynamics. We suggest an iterative scheme for generating a sequence of pulses which create an echo without an external permanent field. The approximation to the ideal echo improves with the sequence length.

PACS numbers: 76.60.Lz

I. INTRODUCTION

The use of echoes is a standard technique in spin resonance experiments[1], where a large number of echo pulse sequences are used. Echo sequences typically consist of a number of high-frequency pluses that induce controlled rotations on the precessing spin. The objective is to remove or reduce dephasing due to inhomogeneities in the external magnetic field which give a variation of the precession rate across the sample (known as inhomogeneous broadening). Since any two-level quantum system can be mapped on a spin-$\frac{1}{2}$ in a magnetic field, similar operations can be performed on any such system, including a qubit. For example, there are recent experiments on superconducting qubits[2,3] (were it is called charge echo, since the physical degree of freedom is two charge states of a superconducting grain) and electron spins in quantum dots[4]. Common to the existing echo techniques is the restriction that in order to apply the proper sequence of control rotations one must know the direction of the magnetic field. In other words, the sequences are able to remove dephasing due to inhomogeneities in the field amplitude, but not the field direction (the randomness in the direction of the magnetic field is indeed negligible if a strong external field is applied). Starting with the experiments[5] that showed long spin coherence times of electrons in bulk GaAs there has been interest in using electrons confined in quantum dots as qubits. It appears that the limiting decoherence factor in these systems is the hyperfine interaction between the electron and nuclear spins[6]. As long as no special preparation is made, the nuclear spin system will give rise to an effective random magnetic field seen by the electron. This field will be random both in strength and direction, and in the absence of an external strong permanent magnetic field the usual echo sequences can not be applied. Because the response of the nuclear spin system is much slower than the precession of the electron spin we can in a certain approximation assume the nuclear field to be constant. Motivated by this we will in this paper investigate the problem of finding an echo sequence that is applicable in the case of a constant effective field with an arbitrary and unknown magnitude and direction. The goal of this sequence is to return the spin to its starting position at the end of the sequence, independently of the magnetic field.

II. FORMULATION OF THE PROBLEM

Consider a spin-$\frac{1}{2}$ particle precessing in a constant magnetic field. Let $\mathbf{n}$ be the unit vector along the precession axis and ω be the precession angular frequency. The rotation of the spin state during time τ is then given by the unitary operator

$$U = \cos\chi I + i \sin\chi n_i \sigma_i, \tag{1}$$

where I and σ_i are the identity and Pauli matrices, and $\chi = \omega\tau/2$. The π rotations about the coordinate axes are denoted $X = i\sigma_x$, $Y = i\sigma_y$, and $Z = i\sigma_z$.

The simplest form of a spin echo is for the situation where the spin is precessing in a field with known direction, say along the z-axis, but with unknown magnitude. Then $n_x = n_y = 0$ and $n_z = 1$, while χ is arbitrary. The usual echo sequence then consists in waiting for the time τ, applying an X rotation, waiting time τ and applying a final X rotation. The success of the procedure is expressed by the fact that $XUXU = -I$ is the identity (up to a global sign, which is unimportant), independently of U. By symmetry, the same is true if X is replaced by a π rotation around any axis in the xy-plane (perpendicular to the precession axis). In particular we have $YUYU = -I$.

We want to extend this to the case were $\mathbf{n}$ is not known. That is, we want to find a sequence of control rotations $A, B, C, \ldots, F$ such that

$$FU \cdots CUBUAU = I \tag{2}$$

for any U. We make the following assumptions: i) The control pulses are effectively instantaneous, meaning they can be performed in a time much less than the precession period. ii) The external magnetic field is unknown but constant, so that the precession operator U does not change in time. Note that we have written Eq. (2) as if

the time τ between the pulses is fixed. Different time intervals between the pulses are achieved by choosing some of the $A, B, C, \ldots, F$ to be the identity. This is sufficient if all intervals are integer multiples of a smallest unit. Intervals with irrational ratios will require a more general form than Eq. (2). In this paper we only use equal intervals.

III. ITERATED MAPPINGS

As was explained above, if we know the direction $\mathbf{n}$ of the external field we can create an echo by applying two π pulses around any axis perpendicular to $\mathbf{n}$. The sequence $XUXU$ can be used if $n_x = 0$, and similarly we may use $YUYU$ if we know that $n_y = 0$. Consider the longer sequence $XUYUXUYU$ for which, using the general U of Eq. (1), we get

$$\begin{aligned} & XUYUXUYU \\ & = (8n_x^2 n_y^2 \sin^4\chi - 1)I - 8in_x n_y^2 \sin^3\chi\cos\chi\sigma_x \\ & - 8in_x n_y^2 \sin^4\chi\sigma_y - 4in_x n_y \sin^2\chi(1 - 2n_y^2\sin^2\chi)\sigma_z . \end{aligned} \tag{3}$$

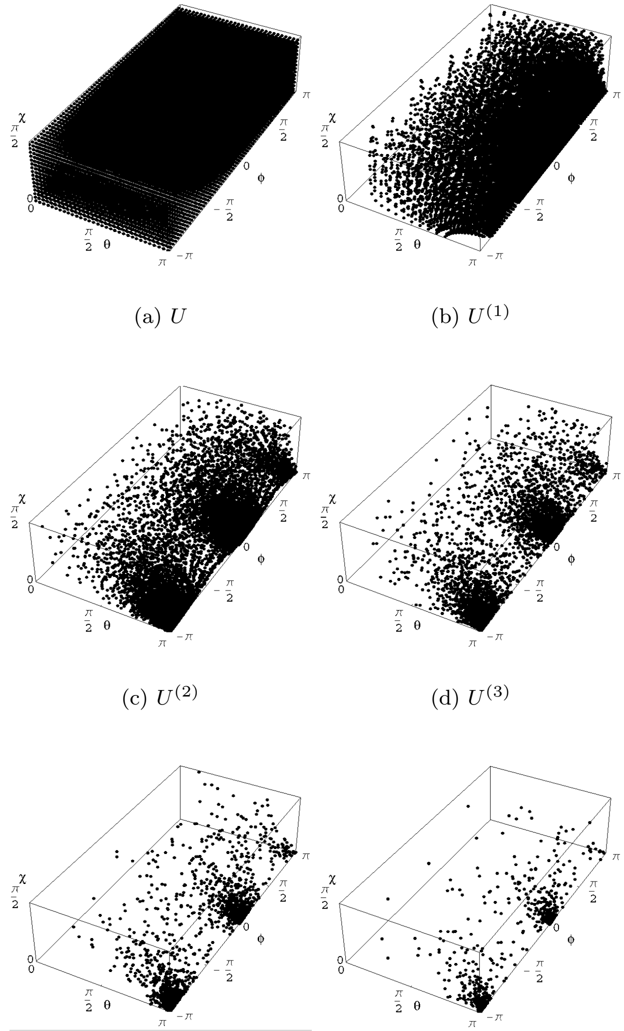
We see that $XUYUXUYU = -I$ for either $n_x = 0$ or $n_y = 0$. This may seem like a small gain, but this sequence is the key to the full solution. Let us construct the mapping

$$U \to T(U) = XUYUXUYU. \tag{4}$$

The idea is that $T(U)$, being composed of the arbitrary U and the fixed control rotations, will be "less arbitrary" than the original U. Iterating this mapping we then construct the set $U^{(1)} = T(U)$, $U^{(2)} = T(U^{(1)})$, ... of pulse sequences that will be better and better approximations to the identity. Since $T(U^{(n)})$ contains $U^{(n)}$ four times the time needed for the sequence $U^{(n)}$ is $4^n\tau$. That is, the length of the sequence grows exponentially in n. This is of course unfortunate as the sequences quickly will become impractically long. However we will see below that for a large portion of the space of parameters determining U, a few iterations n are sufficient to reach a good approximation to the identity. The domain of "bad" parameters shrinks exponentially with the increase of n.

To illustrate this we represent a rotation by the polar angle θ and azimuth angle ϕ of the rotation axis (so that $n_x = \sin\theta\cos\phi$, $n_x = \sin\theta\sin\phi$ and $n_z = \cos\theta$) and the rotation angle χ. Figure 1 shows the evolution of the parameters (θ, ϕ, χ) upon successive transformations given by Eq. (4). The set of initial rotations fills this space uniformly, see Fig. 1 (a). Figures 1 (b)-(f) show the successive iterations $U^{(1)}$ - $U^{(5)}$ of these points, illustrating the convergence of the mapping.

From Eq. (3) we can write down explicit formulas for the parameters $(\mathbf{n}', \chi')$ of $U' = T(U)$ in terms of $(\mathbf{n}, \chi)$,



(a) U (b) $U^{(1)}$ (c) $U^{(2)}$ (d) $U^{(3)}$ (e) $U^{(4)}$ (f) $U^{(5)}$

FIG. 1: (a) A set of rotations U filling uniformly the space (θ, ϕ, χ) of parameters. Each point corresponds to a certain rotation U. (b)-(f) successive iterations of the mapping T(U) showing the convergence to the fixed point $U = -I$.

$$\begin{aligned} \sin\chi' &= 4n_x n_y \sin^2\chi\sqrt{1 - 4n_x^2 n_y^2 \sin^4\chi}, \\ n_x' &= -\frac{2n_y \sin\chi\cos\chi}{\sqrt{1 - 4n_x^2 n_y^2\sin^4\chi}}, \\ n_y' &= -\frac{2n_y n_z \sin^2\chi}{\sqrt{1 - 4n_x^2 n_y^2\sin^4\chi}}, \\ n_z' &= -\frac{1 - 2n_y^2\sin^2\chi}{\sqrt{1 - 4n_x^2 n_y^2\sin^4\chi}}. \end{aligned}$$

We see that $\sin\chi = n_x = n_y = 0$, $n_z = 1$ is a fixed point,

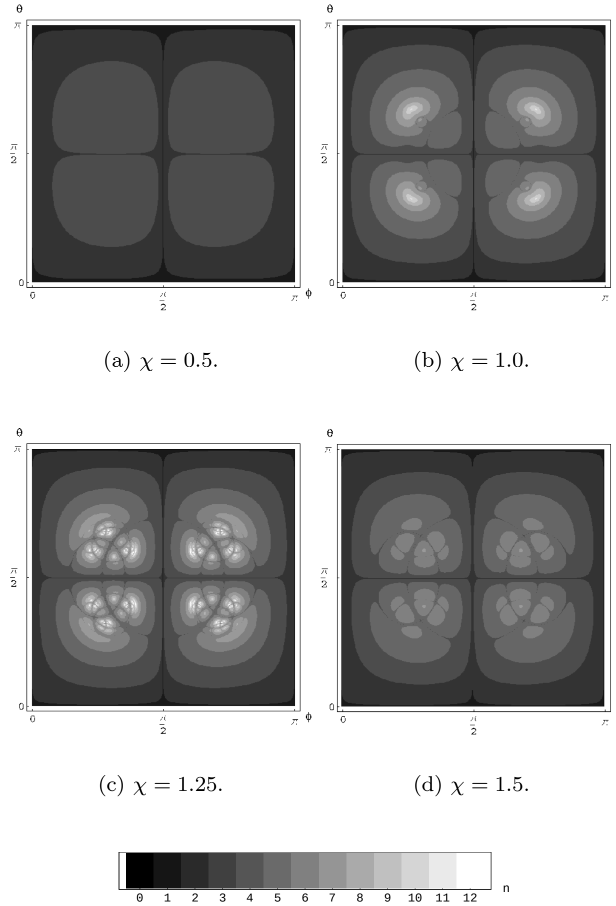
and this represents the identity operator. The stability of the fixed point can be analyzed by expanding close to the fixed point in the independent small quantities $\epsilon_s = \sin\chi/\sqrt{2}$, $\epsilon_x = n_x/\sqrt{2}$ and $\epsilon_y = n_y/2$. We get

$$\begin{aligned}
\epsilon_s' &= \epsilon_x \epsilon_y \epsilon_s^2, \\
\epsilon_x' &= -\epsilon_y \epsilon_s, \\
\epsilon_y' &= -\epsilon_y \epsilon_s^2.
\end{aligned}$$

were the primed quantities refer to the transformed rotation U'. It is clear that if $\epsilon_s, \epsilon_x, \epsilon_y$ are small quantities, then $\epsilon_s', \epsilon_x', \epsilon_y'$ are even smaller and the fixed point is locally stable.

IV. CONVERGENCE PROPERTIES

We studied numerically the convergence of the mapping $T(U)$ for points that are not close to the fixed point. Looking at Figure 1 we see that although most points converge to the vicinity of the fixed point in a few iterations, there are some points that do not converge fast. To get a better understanding we do as follows. Let us choose some initial rotation angle χ and for each point in the (θ, ϕ) plane construct the sequence $U^{(1)}$, $U^{(2)}$, ... stopping when $U^{(n)}$ is within a specified distance from the identity (we used the stopping criterion $\chi^{(n)} < 10^{-3}$). Figure 2 shows the (θ, ϕ) plane for various χ, the shades of gray representing the number n of iterations needed for convergence. We see that for small χ convergence is fast and the pattern is simple, but for larger initial χ the pattern of convergence time is quite complex, and that there exist "hard" points, i.e. initial rotations $(\mathbf{n}, \chi)$ that need a large number n if iterations to converge.



(a) $\chi = 0.5$.

(b) $\chi = 1.0$.

(c) $\chi = 1.25$.

(d) $\chi = 1.5$.

FIG. 2: The levels of gray indicate the number n of iterations that is needed to reach a rotation $U^{(n)}$ with the rotation angle $\chi^{(n)} < 10^{-3}$. The darker shade means faster convergence according to the scale at the bottom.

To estimate the fraction of initial rotations that need a certain number of iterations to converge to the fixed point we do the following. We start with an ensemble of random initial rotations U characterized by a unit vector $\mathbf{n}$ distributed uniformly on a sphere and by an angle χ distributed uniformly from in the inerval $[0, \pi]$. We proceed with the iterations, Eq. (4), until $\chi^{(n)} < 10^{-3}$ is reached. The logarithm of the fraction p_n of initial rotations U that needs n iterations to converge is shown in Figure 3 as a function of n. We find that about 74% of the initial rotations will converge after applying the third iterate $U^{(3)}$ and about 91% do so after the iterate $U^{(4)}$. Observe that except for the first points, all points fall on a straight line which means that the fraction p_n of "difficult" initial rotations decays exponentially with the number of iterations,

$$p_n \propto e^{-\alpha n}, \qquad \alpha \approx 1.1. \tag{5}$$

If k is the number of repetitions of U in $T(U)$ (for the mapping (4) we have $k = 4$), the total time of the sequence $U^{(n)}$ is $t = \tau k^n$, where τ is the time between the pulses. The relation (5) can then also be written as the fraction $p(t)$ of the initial rotations which did not converge till time t,

$$p(t) \propto \left(\frac{t}{\tau}\right)^{-\beta}, \qquad \beta = \frac{\alpha}{\ln k}. \tag{6}$$

The relations Eqs. (5) and (6) are established by running a simulation with the use of a specific convergence criterion, $\chi^{(n)} < 10^{-3}$. Changing the criterion affects the proportionality coefficients in these relations, but does not change the values of exponents α and β.

Let M_n denote the set of initial rotations that do not converge in n iterations. [As an example, the domain M_9 of "difficult" initial rotations (according to the criterion $\chi < 10^{-3}$) is presented in Fig. 4]. The sequence of sets M_n is such that each set is contained in the previuos one, $M_n \subset M_{n-1}$, and the fraction of points p_n in Figure 3 is proportional to the difference in the volumes of M_{n+1} and M_n. The set $M = \bigcap_{n=0}^{\infty} M_n$ of "infinitely hard" points appears to be a fractal with fractal dimension $D \approx 1.5$ according to the box-counting algorithm we used.

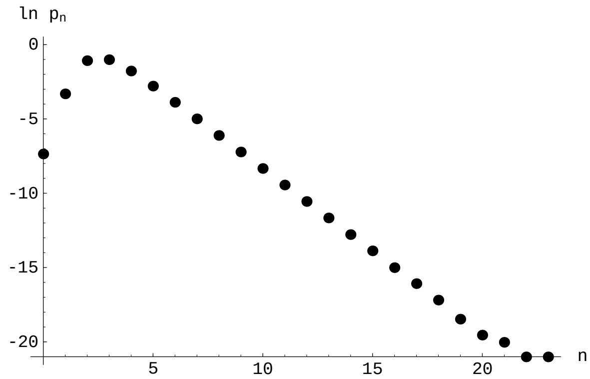


FIG. 3: The fraction p_n of the initial random set of rotations which did not reach the condition $\chi < 10^{-3}$ till the n-th iteration.

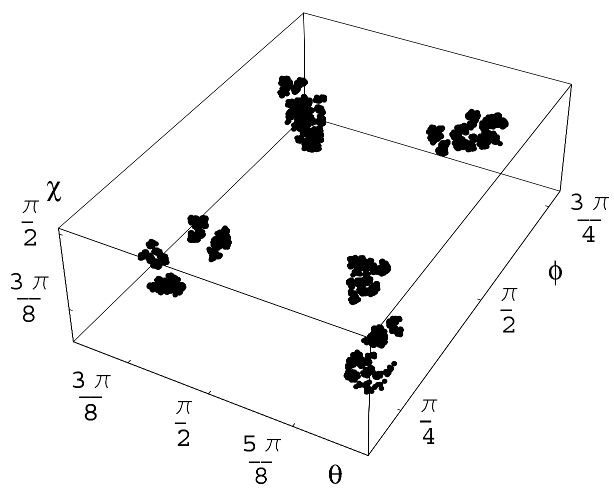


FIG. 4: The set M_9 of initial rotations that do not converge in nine iterations, according to the criterion $\chi < 10^{-3}$.

V. DISCUSSION

We considered the possibility of constructing an echo pulse sequence that does not require applying a high permanent field to the system. The advantage of such a method is that between the pulses the dynamics of the system is not influenced by external perturbations. In the context of ESR of a quantum dot, this method may help distinguishing between different effects of the hyperfine interaction: this interaction creates some random effective magnetic field acting on the electron spin, but may also lead to electron spin relaxation. The absence of the external permanent field in this problem is crucial, as its application definitely suppresses the spin relaxation part of the hyperfine interaction effects[6].

We have provided a solution to the general echo problem in terms of a set of longer and longer pulse sequences that give successively better approximations to the identity operator. We have tested several longer sequences *e.g* $T(U) = ZUXUYUXUZUXUYUXU$. In all cases we found that the exponent β in Eq. (6) is independent of the particular mapping chosen. Whether this represents some inherent property of the problem or only is the case for the limited class of mappings we have studied is not known. For example, we have not studied sequences with control pulses other than π rotations about the coordinate axes. We have also not ruled out the possibility of a solution that will yield an ideal echo with the help of a finite number of control pulses.

The idea of using iterated mappings to generate pulse sequences has been used in NMR applications[7] but as far as we know this particular problem or the mapping we study was never discussed. The mapping we have used was also proposed in Ref. 8 in the context of dynamical decoupling of a qubit.

Acknowledgments

This work was supported by the Norwegian Research Council via a StorForsk program and by NSF grants DMR 02-37296 and DMR 04-39026 at the University of Minnesota. The hospitality of MPIPKS-Dresden (L.I.G.) is gratefully acknowledged.

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